Nijenhuis-Richardson algebra and Frölicher-Nijenhuis Lie module*†

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Abstract

Not associative Nijenhuis-Richardson graded algebra on universal module over Graßmann algebra of differential forms allows a novel/algorithmic definition of the Frölicher-Nijenhuis Lie $\mathbb{R}$-algebra. Some consequences are derived. The signature of the five-dimensional Frobenius subalgebra of the Nijenhuis-Richardson algebra is calculated.

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Contents

1 Introduction

Frölicher and Nijenhuis in 1956 discovered Lie $\mathbb{R}$-algebra implicit structure on a Graßmann module of vector valued differential forms. More on this was presented in Nijenhuis contribution to Edinburg Congress in 1958. Peter Michor since 1985 together with collaborators published many papers and a monograph [Kolár, Michor, Slovák 1993] deeply investigating all aspects of Frölicher and Nijenhuis Lie bracket. Dubois-Violette and Michor in 1995 found a common generalization of the Frölicher-Nijenhuis bracket and the Schouten bracket for the symmetric algebra of multi-vector fields.

The Frölicher and Nijenhuis Lie module and Lie $\mathbb{R}$-operation found very important applications/interpretations in differential geometry of connections (and in particular the Nijenhuis tensor that describe the curvature of an almost product structure) [Gray 1967, Gancarzewicz 1987, Kocik 1997, Krasil'shchik and Verbovetsky 1998, Wagemann 1998], in algebraic geometry, in cohomology of Lie algebras [Wagemann 1999], in special relativity theory, in Maxwell's theory of electromagnetic field [Fecko 1997, Kocik 1997, Cruz and Oziewicz 2003], in Einstein's gravity theory [Minguzzi 2003], in classical mechanics for symplectic structure [Gruhn and Oziewicz 1983, Gozzi and Mauro 2000, Chavchanidze 2003].

From the point of view of applications there is a need, among other, for the explicit/algorithmic definition/expression for the Frölicher and Nijenhuis Lie operation, such that can be implemented for symbolic program.

In the present note we remaind the basic concepts, and we are proposing a novel/algorithmic explicit definition of the Frölicher and Nijenhuis Lie $\mathbb{R}$-operation in terms of the primary non-associative (Lie-admissible) $\mathcal{F}$-algebra structure on universal Graßmann module of vector-valued differential forms, that was introduced by Nijenhuis and Richardson in a year 1967.

The non-associative Nijenhuis-Richardson primary algebra, that we need in order to define Frölicher and Nijenhuis Lie operation, is a natural extension of the associative algebra of endomorphisms, trace-class $(1,1)$-fields, to algebra of $(\text{any},1)$-fields with generalized Graßmann-valued 'trace'.

The main objective of this note is rethink the basic concepts, introduce a novel/algorithmic definition of the differential Frölicher and Nijenhuis Lie Graßmann-module, presentation some consequences of this definition, and provide a detailed proofs of some statements that otherwise it is hard to find in available literature.

The Nijenhuis-Richardson not associative algebra possess the associative subalgebra, that is the Frobenius algebra. For Frobenius algebra we refer to [Frobenius 1903, Curtis & Reiner 1962, Kauffman 1994, Voronov 1994, Beidar et al. 1997, Kadison 1999, Baez 2001, Caenepeel et al. 2002]. In the last Sections we briefly define the Frobenius algebra, and initiate study of the five-dimensional Frobenius associative subalgebra of the Nijenhuis-Richardson not associative algebra.

References include all known to us publications related to the subject of the present paper, even so we do not made comments about some of them.

Some Notation

$\mathcal{F}$	- denotes the associative, unital and commutative ring, e.g. $\mathbb{R}$-algebra.
$\operatorname{der}_{\mathbb{R}} \mathcal{F}$ $\equiv \operatorname{der}_{\mathbb{R}}(\mathcal{F}, \mathcal{F})$	-denotes the Lie $\mathcal{F}$-module of the derivations, Lie $\mathcal{F}$-modul of the vector fields.
$M = {}_{\mathcal{F}}M$	- denotes the projective $\mathcal{F}$-module of the differential 1-forms (the Pfaffian forms), $\dim_{\mathcal{F}} M < \infty$, with a derivation $d \in \operatorname{der}_{\mathbb{R}}(\mathcal{F}, M)$. Then $M = (\operatorname{der}_{\mathbb{R}} \mathcal{F})^* \equiv \operatorname{mod}_{\mathcal{F}}(\operatorname{der}_{\mathbb{R}} \mathcal{F}, \mathcal{F})$.
M^* $(-)^{AB}$	- denotes 'dual of dual' $\mathcal{F}$-modul of the vector fields, $M^* \equiv \operatorname{mod}_{\mathcal{F}}(M, \mathcal{F}) = (\operatorname{der}_{\mathbb{R}} \mathcal{F})^{**} \simeq \operatorname{der}_{\mathbb{R}} \mathcal{F}$. - is an abbreviation for $(-1)^{(\operatorname{grade} A)(\operatorname{grade} B)}$.

2 Universal Graßmann module

In the sequel the Graßmann $\mathcal{F}$-factor-algebra of differential multi-forms is denoted by $M^\wedge \equiv M^\otimes/I$, where $I < M^\otimes$ is an ideal in a free tensor $\mathcal{F}$-algebra, generated by $\alpha \otimes \alpha \ \forall\, \alpha \in M$. A left $M^\wedge$-module $M^\wedge \otimes_{\mathcal{F}} M^* \simeq \operatorname{mod}_{\mathcal{F}}(M, M^\wedge)$ is said to be an M^*-universal Graßmann-module, known variously as the module of 'vector-valued differential forms' or module of 'vector-forms'.

An $\mathbb{R}$-linear or $\mathcal{F}$-linear homogeneous endomorphism $D \in \operatorname{End}(M^\wedge)$ with $\operatorname{grade} D \in \mathbb{Z}$, is said to be a $\mathbb{Z}_2$-graded derivation (skew derivation, anti-derivation), $D \in \operatorname{der}(M^\wedge)$, if the graded Leibniz axiom holds. Derivation is said to be *algebraic* if $D|\mathcal{F} = 0$.

A $\mathbb{Z}$-graded Lie $\mathcal{F}$-algebra of $\mathcal{F}$-derivations of the Graßmann $\mathcal{F}$-algebra, $\operatorname{der}_{\mathcal{F}}(M^\wedge)$, is a left $M^\wedge$-module. We are going to describe an $M^\wedge$-module isomorphism that Nijenhuis and Richardson in 1967 extended to isomorphism of graded commutators (actually this is an isomorphism of Gerstenhaber algebras, see Lemma 3.5, etc),

$$i \in \operatorname{mod}_{M^\wedge}(M^\wedge \otimes_{\mathcal{F}} M^*, \operatorname{der}_{\mathcal{F}}(M^\wedge)). \tag{2.1}$$

Every derivation of a Graßmann algebra $D \in \operatorname{der}(M^\wedge \equiv M^\otimes/I)$ is uniquely determined by values of D on generating $\mathbb{R}$-algebra $\mathcal{F}$ and on $\mathcal{F}$-module M: $D|\mathcal{F} \in \operatorname{der}_{\mathbb{R}}(\mathcal{F}, M^\wedge)$ and $D|M \in \operatorname{der}_{\mathbb{R}}(M, M^\wedge)$, if and only if $(D|M)^\otimes I \subset I$. Therefore a $\mathbb{Z}$-homogeneous derivation D with a $\operatorname{grade} D \leq -2$ must be the trivial zero derivation.

A $\mathcal{F}$-module map $p \in \operatorname{mod}_{\mathcal{F}}(M, M^\wedge)$ lifts to the unique $\mathbb{Z}_2$-graded $\mathcal{F}$-derivation i_p with $\operatorname{grade}(i) = 0$, such that $i_p|\mathcal{F} = 0$ and $i_p|M = p$,

$$\operatorname{mod}_{\mathcal{F}}(M, M^\wedge) \simeq M^\wedge \otimes_{\mathcal{F}} M^* \ni p \quad \xmapsto{i} \quad i_p \in \operatorname{der}_{\mathcal{F}}(M^\wedge), \tag{2.2}$$

Let $\alpha, \beta \in M^\wedge$, $X \in M^* \simeq \operatorname{der}_{\mathbb{R}} \mathcal{F}$ and $p \equiv \alpha \otimes_{\mathcal{F}} X \in (M^\wedge \otimes_{\mathcal{F}} M^*)$. We abbreviate $\beta \wedge p = \beta p$. Then [e.g Dubois-Violette and Michor 1995]

$$\begin{gathered} i(\alpha \otimes_{\mathcal{F}} X) \equiv e_\alpha \circ i_X, \quad i_X \in \operatorname{der}_{\mathcal{F}}(M^\wedge), \quad i_{\alpha p} = \alpha\, i_p, \\ \operatorname{grade}(e_\alpha \circ i_T) = -1 + \operatorname{grade} \alpha. \end{gathered} \tag{2.3}$$

If $p \in \operatorname{der}_{\mathcal{F}}(M^\wedge)$, then $(i \circ |M)p = p$. Therefore the restriction '$|M$' is the inverse of (2.1)-(2.2)-(2.3), $i^{-1} = |M$, and there is a bijection,

$$\operatorname{der}_{\mathcal{F}}(M^\wedge) \xrightarrow{|M = i^{-1}} M^\wedge \otimes_{\mathcal{F}} M^*. \tag{2.4}$$

Example. A vector field $T \in M^* \simeq \operatorname{mod}_{\mathcal{F}}(M, \mathcal{F})$ lifts to an algebraic derivation $T \mapsto i_T \in \operatorname{der}_{\mathcal{F}}(M^\wedge)$ with $\operatorname{grade}(i_T) = -1$.

2.1 Nijenhuis-Richardson algebra

Consider $p, q \in M^\wedge \otimes_{\mathcal{F}} M^* \simeq \operatorname{mod}_{\mathcal{F}}(M, M^\wedge)$. Under this identification Nijenhuis and Richardson in 1967 defined not associative $\mathcal{F}$-algebra as follows.

2.1 Definition (Nijenhuis-Richardson algebra)**.** Let $\alpha, \beta \in M^\wedge$.

$$\begin{gathered} \{\operatorname{mod}_{\mathcal{F}}(M, M^\wedge)\} \otimes_{\mathcal{F}} \{\operatorname{mod}_{\mathcal{F}}(M, M^\wedge)\} \longrightarrow \{\operatorname{mod}_{\mathcal{F}}(M, M^\wedge)\}, \\ p \otimes_{\mathcal{F}} q \longmapsto pq \equiv (ip) \circ q \in \{\operatorname{mod}_{\mathcal{F}}(M, M^\wedge)\}. \end{gathered}$$

If $\quad p = \alpha \otimes_{\mathcal{F}} P \quad$ and $\quad q = \beta \otimes_{\mathcal{F}} Q, \quad$ then $\quad pq = (\alpha \wedge (i_P \beta)) \otimes_{\mathcal{F}} Q.$

Clearly $(\alpha p)q = \alpha(pq)$. However for $\alpha \in M^\wedge$, $p\alpha \equiv (ip)\alpha$ and every vector valued differential form is his own $M^\wedge$-module derivation, e.g. [Dubois-Violette and Michor 1994],

$$p(\alpha q) = (p\alpha)q + (-)^{p\alpha} \alpha(pq), \tag{2.5}$$

$$\{p \otimes_{\mathcal{F}} (\alpha q)\} = (i_p \alpha) q + (-)^{p\alpha} \{p \otimes_{\mathcal{F}} q\}. \tag{2.6}$$

The Nijenhuis-Richardson $\mathbb{Z}$-graded $\mathcal{F}$-algebra is not associative, not unital, and not commutative,

$$(pq)r \equiv i(i_p \circ q) \circ r \quad \neq \quad i_p \circ (i_q \circ r) \equiv p(qr), \tag{2.7}$$

$$i_{pq} = i_p \circ i_q + i_{p \wedge q} \quad \in \operatorname{der}(M^\wedge). \tag{2.8}$$

$$\text{If } \operatorname{grade} q = -1 \quad \text{then } \forall\, p, \quad pq = 0. \tag{2.9}$$

3 Leibniz/Loday and Gerstenhaber algebra

Let $\mathcal{F}$ be a ring and A be $\mathcal{F}$-bimodule. A category of $\mathcal{F}$-bimodules is a monoidal abelian category.

3.1 Definition (Leibniz/Loday algebra, Loday 1993)**.** A pair of binary operations/morphisms, $\cap$ and $[\cdot,\cdot]$, is said to be the Leibniz/Loday algebra if

$$[\cdot,\cdot] \in \operatorname{der}\cap, \qquad \text{carrier} \xrightarrow{[\cdot,\cdot]} \operatorname{der}\cap. \tag{3.1}$$

A graded Leibniz algebra is a pair of homogeneous binary operations $\cap$ and $[\cdot,\cdot]$ on a $\mathbb{Z}$-graded object/carrier such that $\forall\, a, b \in$ carrier, $[a \equiv [a,\cdot] \in \operatorname{der}\cap$,

$$([a) \circ \cap_b = \cap_{[a,b]} + (-1)^{(a+[\cdot,\cdot])(b+\cap)} \cdot \cap_b \circ ([a) \quad \in \operatorname{End}\ A. \tag{3.2}$$

3.2 Definition (Gerstenhaber algebra)**.** The $\mathbb{Z}$-graded Leibniz algebra $(\cap,[\cdot,\cdot])$ is said to be the graded Poisson algebra or the graded Gerstenhaber algebra if

$$\operatorname{grade}[\cdot,\cdot] + \operatorname{grade}\cap = \begin{cases} \text{even} & -\text{the Poisson algebra,} \\ \text{odd} & -\text{the Gerstenhaber algebra.} \end{cases}$$

Definition 3.2 [Oziewicz and Paal 1995] generalize the Gerstenhaber [1963] structure carried by the Hochschild cohomology of an associative algebra $\cap$. In this definition both binary operations need not to be graded commutative, $\cap$ need not to be associative, and $[\cdot,\cdot]$ need not to be Lie-admissible. However a crossing $2 \mapsto 2$ needs to be the Artin braid [Oziewicz, Różański and Paal 1995]. A concept of the Lie-Cartan pair introduced by Jadczyk and Kastler [1987, 1991] is a generalization of Leibniz algebra to pair of objects, it is a two-sorted Leibniz/Loday algebra.

3.3 Graded commutator. Let A, B, C be $\mathbb{R}$- or $\mathcal{F}$- linear $\mathbb{Z}$-homogeneous graded endomorphisms $A, B, C \in \operatorname{End}(M^\wedge)$. We abbreviate $(-1)^{(\operatorname{grade} A)(\operatorname{grade} B)}$ to $(-)^{AB}$. The graded commutator (bracket) needs the Koszul rule of signs

$$\begin{gathered}\{A \otimes_{\mathbb{R}/\mathcal{F}} B\} \equiv A \circ B - (-)^{AB} B \circ A, \\ \operatorname{grade}\{A \otimes B\} = \operatorname{grade}\{\cdot,\cdot\} + \operatorname{grade} A + \operatorname{grade} B.\end{gathered} \tag{3.3}$$

Thanks associativity of a composition this is an example of the $\mathbb{Z}$-graded Poisson algebra

$$\{A \otimes (B \circ C)\} = \{A \otimes B\} \circ C + (-)^{AB} \cdot B \circ \{A \otimes C\}. \tag{3.4}$$

An associative $\mathbb{Z}_2$-graded $\mathbb{R}$- and $\mathcal{F}$-algebra End $(M^\wedge)$ with the above commutator is a $\mathbb{Z}$-graded Poisson $\mathcal{F}$-algebra and a Lie ring. The Jacobi identity is a consequence of (3.4),

$$\{A \otimes \{B \otimes C\}\} = \{\{A \otimes B\} \otimes C\} + (-)^{AB}\{B \otimes \{A \otimes C\}\}.$$

3.4 Lemma (Lie super algebra of derivations)**.** *Let* $A, B \in \operatorname{der}(M^\wedge)$. *Then* $\{A \otimes B\} \in \operatorname{der}(M^\wedge)$.

Proof. Every commutator (graded or 'not graded' with trivial grading) is an inner derivation in the Lie admissible ring of an ($\mathbb{Z}$-graded) abelian group-endomorphisms. This implies that the commutator of derivations (of a ring) is again the derivation.

Therefore the space of derivations is a $\mathbb{Z}_2$-graded Lie algebra (*i.e.* a super-algebra), a sub-algebra of End $(M^\wedge)$ with $\{\cdot \otimes \cdot\} \equiv \{,\}$.

Independently one can check Lemma 3.4 by direct computation. In particular $\{D, D\} = (1 - (-)^D) \cdot D^2$, therefore for a derivation D, a map D^2 is again a nontrivial derivation if grade $D =$ odd. □

3.5 Lemma (Nijenhuis and Richardson 1967)**.** *Let* $p, q \in M^\wedge \otimes_{\mathcal{F}} M^*$. *The* $\mathcal{F}$*-module isomorphism* (2.1)-(2.3) *is a graded Lie* $\mathcal{F}$*-algebra map:*

$$\{i_p \otimes_{\mathcal{F}} i_q\} = i\{p \otimes_{\mathcal{F}} q\} \quad \in \operatorname{der}_{\mathcal{F}}(M^\wedge). \tag{3.5}$$

Proof. An equality of algebraic derivations must be verified on restriction $i^{-1} \equiv |M$. □

4 Frölicher and Nijenhuis decomposition

4.1 Universal property of derivation

The derivation $d \in \operatorname{der}_{\mathbb{R}}(\mathcal{F}, M)$ has the universal property: for $D \in \operatorname{der}_{\mathbb{R}}(\mathcal{F}, M^\wedge)$, there is the unique $\mathcal{F}$-module map, $j_D \in \operatorname{mod}_{\mathcal{F}}(M, M^\wedge)$, such that $D = j_D \circ d$, grade $j = -1$,

$$\begin{tikzcd} \mathcal{F} \arrow[r, "d"] \arrow[d, equal] & M \arrow[d, "j_D"] \\ \mathcal{F} \arrow[r, "D"] & M^\wedge \end{tikzcd}$$

$$\operatorname{der}_{\mathbb{R}}(\mathcal{F}, M^\wedge) \xrightarrow{j} \operatorname{mod}_{\mathcal{F}}(M, M^\wedge) \xrightarrow{i} \operatorname{der}_{\mathcal{F}}(M^\wedge). \tag{4.1}$$

In particular $d = j_d \circ d \;\Rightarrow\; j_d = \mathrm{id}_M$. The grade operator is a derivation,

$$\mathrm{End}_{\mathcal{F}} M = \mathrm{mod}_{\mathcal{F}}(M, M) \ni \mathrm{id}_M \overset{i}{\longmapsto} \mathrm{grade} \equiv i_{\mathrm{id}} \in \mathrm{der}_{\mathcal{F}}(M^{\wedge}), \tag{4.2}$$

$$\{(i \circ j)\, d, d\} = d. \tag{4.3}$$

From the universal property of $d \in \mathrm{der}_{\mathbb{R}}(\mathcal{F}, M)$ it follows the $\mathcal{F}$-module isomorphism of the vector fields, $\mathrm{der}_{\mathbb{R}} \mathcal{F} \equiv \mathrm{der}_{\mathbb{R}}(\mathcal{F}, \mathcal{F})$ with the $\mathcal{F}$-dual $\mathcal{F}$-module, $M^* \equiv \mathrm{mod}_{\mathcal{F}}(M, \mathcal{F}) \equiv \mathcal{F}^M$. Let $T \in \mathrm{der}\, \mathcal{F}$, then

$$\forall\, f \in \mathcal{F}, \quad Tf \equiv (df)T \equiv j_T df \in \mathcal{F},$$

$$\mathrm{der}_{\mathbb{R}}(\mathcal{F}, \mathcal{F}) \xrightarrow{\;j\;} M^*$$

$$\mathrm{der}_{\mathbb{R}}(\mathcal{F}, \mathcal{F}) \xleftarrow{\;d^*\;} M^*.$$

Therefore $\mathrm{der}_{\mathbb{R}} \mathcal{F} \ni T = j_T \circ d = d^*(j_T) = (d^* \circ j)T$.

4.2 Lie-Ślebodziński derivation

The Graßmann-Hopf $\mathcal{F}$-algebra $M^{\wedge}$ with the unique lifted graded differential, $d \in \mathrm{der}_{\mathbb{R}}(M^{\wedge})$, $\mathrm{grade}\, d = +1$, $d^2 = 0 \in \mathrm{der}_{\mathbb{R}}(M^{\wedge})$, is said to be the differential $\mathbb{N}$-graded algebra (DGA), de Rham complex. The following $\mathbb{R}$-derivation with $\mathrm{grade}\, \mathcal{L} = +1$ is said to be the (right/left) Lie-Ślebodziński derivation of the endomorphism algebra End ,

$$\mathcal{L}^{r/l} \in \mathrm{der}^{r/l}_{\mathbb{R}}(\mathrm{End}_{\mathbb{R}}(M^{\wedge})) \equiv \mathrm{der}^{r/l}_{R}(\circ),$$

$$\begin{aligned} \mathrm{End}_{\mathbb{R}}(M^{\wedge}) \ni A &\xrightarrow{\;\mathcal{L}^r\;} \mathcal{L}^r_A \equiv \{A, d\} \in \mathrm{End}_{\mathbb{R}}(M^{\wedge}), \\ \mathrm{der}_{\mathbb{R}}(M^{\wedge}) \ni p &\xrightarrow{\;\mathcal{L}^r\;} \mathcal{L}^r_p \equiv \{p, d\} \in \mathrm{der}_{\mathbb{R}}(M^{\wedge}), \end{aligned} \tag{4.4}$$

$$d^2 = 0 \quad \Longrightarrow \quad \mathcal{L}^2 = 0. \tag{4.5}$$

The last implication follows from graded Jacobi identity $\mathcal{L}^2 A = \{A, d^2\}$.

Let $A \in \mathrm{End}\,(M^{\wedge})$ be a $\mathbb{Z}$-graded $\mathcal{F}$- or $\mathbb{R}$-map, and $f \in \mathcal{F}$. Then

$$\mathcal{L}^r_A \equiv \{A, d\} \equiv A \circ d - (-)^A \cdot d \circ A \equiv (-)^{1+A} \mathcal{L}^l_A \quad \in \mathrm{End}_{\mathbb{R}}(M^{\wedge}), \tag{4.6}$$

$$\mathcal{L}_f \equiv \{f, d\} = -e_{df} \equiv -(df) \wedge \ldots, \tag{4.7}$$

$$\mathcal{L}^r_{A \circ B} = (-1)^B \mathcal{L}^r_A \circ B + A \circ \mathcal{L}^r_B. \tag{4.8}$$

For a multivector fields $X, Y \in M^{*\wedge}$, $i_{X\wedge Y} = i_Y \circ i_X \in \operatorname{End}(M^\wedge)$ (for grade $X \geq 2$, $i_X \notin \operatorname{der}(M^\wedge)$), and $\mathcal{L}_X \equiv \{i_X, d\} \in \operatorname{End}_{\mathbb{R}}(M^\wedge)$ [Tulczyjew 1974].

For a 1-vector field, $X \in \operatorname{der}_{\mathbb{R}} \mathcal{F} \equiv \operatorname{der}_{\mathbb{R}}(\mathcal{F}, \mathcal{F}) \simeq M^* \ni jX$, lifted to $\mathcal{F}$-derivation of the Graßmann algebra $(i \circ j)X \in \operatorname{der}_{\mathcal{F}}(M^\wedge)$, the 0-grade directional $\mathbb{R}$-derivation along a 1-vector field $X \in \operatorname{der} \mathcal{F}$, $\mathcal{L}_X \equiv \{(i \circ j)X, d\} \in \operatorname{der}_{\mathbb{R}}(M^\wedge)$, was invented by Ślebodziński [1931]. For $X \in \operatorname{der} \mathcal{F}$, and for $f \in \mathcal{F}$, we have

$$\mathcal{L}_X \equiv \mathcal{L}_{(i\circ j)X}, \quad (\mathcal{L}^2)X = \{\mathcal{L}_X, d\} = 0, \tag{4.9}$$
$$\mathcal{L}_X f = (i \circ j)_X df = j_X df = Xf. \tag{4.10}$$

The name 'Lie derivation' along the vector field $X \in \operatorname{der} \mathcal{F}$, was introduced by D. van Dantzig (collaborator of Schouten). The Lie-Ślebodziński derivation is implicit in [Cartan 1922].

The Lie-Ślebodziński $M^\wedge$-module graded right/left derivations

$$\mathcal{L}^l_A \equiv \{d, A\} = (-)^{1+A} \mathcal{L}^r_A,$$

possess the following Leibniz expressions for $\alpha \in M^\wedge$ and $q \in M^\wedge \otimes_{\mathcal{F}} M^*$,

$$\begin{aligned} \mathcal{L}^r_{i(\alpha q)} &= (-)^{1+\alpha+q}(d\alpha) \wedge q + \alpha \wedge \mathcal{L}^r_{iq}, \\ \mathcal{L}^l_{i(\alpha q)} &= (d\alpha) \wedge q + (-)^\alpha \alpha \wedge \mathcal{L}^l_{iq}. \end{aligned} \tag{4.11}$$

4.3 Frölicher and Nijenhuis decomposition

In the sequel we use the universal property (4.1), and to simplify notation we write j instead of the composition $j \circ (|M)$. In this convention (4.1) reads

$$\operatorname{der}_{\mathbb{R}}(M^\wedge) \xrightarrow{i \circ j \circ (|\mathcal{F})} \operatorname{der}_{\mathcal{F}}(M^\wedge). \tag{4.12}$$

Theorem 4.3.1 (Frölicher and Nijenhuis 1956). Any $\mathbb{R}$-derivation $D \in \operatorname{der}_{\mathbb{R}}(M^\wedge)$ possess the following unique decomposition

$$D = (\mathcal{L} \circ i \circ j + i \circ j \circ \mathcal{L})D = \{i_{jD}, d\} + i_{j\{D,d\}}. \tag{4.13}$$

Proof. First we need remaind the definitions of 'vector-forms' (4.1),

$$jD, \quad j\mathcal{L}_D \quad \in M^\wedge \otimes_F M^*.$$

For $D \in \mathrm{der}_{\mathbb{R}}(M^\wedge)$, $D|\mathcal{F} \in \mathrm{der}_{\mathbb{R}}(\mathcal{F}, M^\wedge)$. Universality of $d \in \mathrm{der}_{\mathbb{R}}(\mathcal{F}, M)$ gives

$$D|\mathcal{F} \equiv (jD) \circ d, \quad jD = 0 \Longleftrightarrow D|\mathcal{F} = 0, \tag{4.14}$$
$$\mathcal{L}_D|\mathcal{F} \equiv (j\mathcal{L}_D) \circ d. \tag{4.15}$$

The Frölicher and Nijenhuis decomposition (4.13) is an equality of derivations, $D = 0$ iff $D|\mathcal{F} = 0$ and $D|d\mathcal{F} = 0$. We must check that the F-N decomposition (4.13) is an identity on a ring $\mathcal{F}$ and on exact differential one-forms $d\mathcal{F} < M$. □

5 Main definition

Let $A, B \in \mathrm{End}_{\mathcal{F}}(M^\wedge)$, *i.e.* $Af = 0$. Then

$$\mathcal{L}^r_{(A\circ B)}|\mathcal{F} = (A \circ B)|d\mathcal{F}, \quad \mathcal{L}^l_{A\circ B}|\mathcal{F} = \dots$$
$$\{A \otimes_{\mathbb{R}} \mathcal{L}^r_B\}|\mathcal{F} = (A \circ B)|d\mathcal{F}.$$

Set an $M^\wedge$-module map (2.4), $i^{-1} \equiv |M : \mathrm{der}_{\mathcal{F}}(M^\wedge) \longrightarrow M^\wedge \otimes_{\mathcal{F}} M^*$. Let $p, q, pq, qp \in M^\wedge \otimes_{\mathcal{F}} M^*$, where pq is the Nijenhuis-Richardson nonassociative product. We have

$$\mathcal{L}_{i(pq)}|\mathcal{F} = (pq)d|\mathcal{F}, \tag{5.1}$$
$$\{i_p \otimes_{\mathbb{R}} \mathcal{L}_{iq}\}|\mathcal{F} = i_p \circ \mathcal{L}_{iq}|\mathcal{F} = i_p \circ i_q \circ d|\mathcal{F} = (i_p \circ q)d|\mathcal{F}. \tag{5.2}$$

This proves that

$$\mathcal{L}^{r/l}_{i(pq)} - \{i_p \otimes_{\mathbb{R}} \mathcal{L}^{r/l}_{iq}\} \in \mathrm{der}_{\mathcal{F}}(M^\wedge). \tag{5.3}$$

The Frölicher-Nijenhuis differential binary operation on the $\mathbb{R}$-Lie $M^\wedge$-module $\mathrm{der}_{\mathcal{F}}(M^\wedge) \simeq M^\wedge \otimes_{\mathcal{F}} M^*$, is denoted by $[\cdot \otimes_{\mathbb{R}} \cdot]$, with $\mathrm{grade}[\cdot \otimes_{\mathbb{R}} \cdot] = +1$,

$$(M^\wedge \otimes_{\mathcal{F}} M^*) \otimes_R (M^\wedge \otimes_{\mathcal{F}} M^*) \xrightarrow{[\cdot\otimes_{\mathbb{R}}\cdot]} (M^\wedge \otimes_{\mathcal{F}} M^*).$$

5.1 Frölicher-Nijenhuis Lie $M^\wedge$-module. We define the following algorithmic/explicit form of the Frölicher-Nijenhuis $\mathbb{R}$-bracket,

$$(-)^q\, i[p \otimes_{\mathbb{R}} q] \equiv \mathcal{L}^r_{i(pq)} - \{i_p \otimes_{\mathbb{R}} \mathcal{L}^r_{iq}\} \qquad \in \mathrm{der}_{\mathcal{F}}(M^\wedge), \tag{5.4}$$
$$\boxed{(-)^q[p \otimes_{\mathbb{R}} q] \equiv i^{-1}\left(\mathcal{L}^r_{i(pq)} - \{i_p \otimes_{\mathbb{R}} \mathcal{L}^r_{iq}\}\right)} \quad \in M^\wedge \otimes_{\mathcal{F}} M^*. \tag{5.5}$$

In particular if p is an idempotent (with respect to Nijenhuis-Richardson product), $p^2 = p \in M^\wedge \otimes_{\mathcal{F}} M^*$, then grade $p = 0$ and

$$i[p \otimes_{\mathbb{R}} p] = \mathcal{L}^r_{ip} - \{i_p \otimes_{\mathbb{R}} \mathcal{L}^r_{ip}\} = 2 i_p d i_p. \tag{5.6}$$

5.2 Lemma. *The binary* $\mathbb{R}$*-operation* (5.5) *is graded commutative*

$$[p \otimes_{\mathbb{R}} q] = (-1)^{p+q+pq} \cdot [q \otimes_{\mathbb{R}} p]. \tag{5.7}$$

Proof. For $p, q \in M^\wedge \otimes_{\mathcal{F}} M^*$, and for $A, B \in \operatorname{der}_{\mathcal{F}}(M^\wedge)$, we have

$$i\{p \otimes_{\mathcal{F}} q\} = \{i_p \otimes_{\mathcal{F}} i_q\} \quad \Longleftrightarrow \quad i_{pq} - (-)^{pq}\, i_{qp} = \{i_p \otimes_{\mathcal{F}} i_q\}, \tag{5.8}$$

$$\mathcal{L}_{A \otimes_{\mathcal{F}} B} = \{A \otimes_{\mathbb{R}} \mathcal{L}_B\} + (-)^B \{\mathcal{L}_A \otimes_{\mathbb{R}} B\}, \tag{5.9}$$

$$\mathcal{L}_{i(pq)} = (-)^{pq}\, \mathcal{L}_{i(qp)} + \{i_p \otimes_{\mathbb{R}} \mathcal{L}_{iq}\} + (-)^q \{\mathcal{L}_{ip} \otimes_{\mathbb{R}} i_q\}. \tag{5.10}$$

All this implies that

$$\begin{aligned}
(-)^q [p \otimes_{\mathbb{R}} q] &= \mathcal{L}_{i(pq)} - \{i_p \otimes_{\mathbb{R}} \mathcal{L}_{iq}\} && (5.11)\\
&= (-)^{pq}\, \mathcal{L}_{i(qp)} + (-)^q \{\mathcal{L}_{ip} \otimes_{\mathbb{R}} i_q\} && (5.12)\\
&= (-)^{p+pq}\, [q \otimes_{\mathbb{R}} p]. \quad \square && (5.13)
\end{aligned}$$

In order to relate Definition (5.4)-(5.5) with the original *implicit* Definition by Frölicher and Nijenhuis [1956], we need to calculate the Lie-Ślebodziński map on (5.4),

$$\mathcal{L}_{i[p \otimes_{\mathbb{R}} q]} = (-)^{1+q} \{\{i_p \otimes_{\mathbb{R}} \mathcal{L}_{iq}\} \otimes_{\mathbb{R}} d\} = \{\mathcal{L}_{ip} \otimes_{\mathbb{R}} \mathcal{L}_{iq}\}. \tag{5.14}$$

The original, implicit Definition by Frölicher-Nijenhuis is as follows. By the Jacobi identity we have,

$$\begin{gathered}
\mathcal{L} \circ \mathcal{L} = 0 \quad \Longrightarrow \\
\{\mathcal{L}_A \otimes_{\mathbb{R}} d\} = 0 \quad \& \quad \{\{\mathcal{L}_A \otimes_{\mathbb{R}} \mathcal{L}_B\} \otimes_{\mathbb{R}} d\} = 0.
\end{gathered} \tag{5.15}$$

The Frölicher and Nijenhuis decomposition [1956] (4.13) implies that for $A, B \in \operatorname{der}_{\mathcal{F}}(M^\wedge)$ a derivation $[A \otimes_{\mathbb{R}} B] \in \operatorname{der}_{\mathcal{F}}(M^\wedge)$ exists (in an implicit way) such that

$$\mathcal{L}_{[A \otimes_{\mathbb{R}} B]} \equiv \{\mathcal{L}_A \otimes_{\mathbb{R}} \mathcal{L}_B\} \in \operatorname{der}_{\mathbb{R}}(M^\wedge), \tag{5.16}$$

$$[A \otimes_{\mathbb{R}} B] = (-1)^{A+B+AB} \cdot [B \otimes_{\mathbb{R}} A]. \tag{5.17}$$

5.3 Example. If grade $q = -1$ we set $q = X \in M^*$. Then $\forall\, p \in M^\wedge \otimes_{\mathcal{F}} M^*$, $pq = 0 \in M^\wedge \otimes_{\mathcal{F}} M^*$. In this case the Definition (5.4)-(5.5) is simplified

$$i[p \otimes_{\mathbb{R}} X] = \{i_p \otimes_{\mathbb{R}} \mathcal{L}_{iX}\}. \tag{5.18}$$

Evaluating above brackets on exact 1-form $df \in M$, is showing that the Frölicher and Nijenhuis Lie $M^\wedge$-module generalize Lie $\mathcal{F}$-module of the vector fields

$$[p \otimes_{\mathbb{R}} X]df = i_p d(Xf) - (\mathcal{L}_{iX})pdf. \tag{5.19}$$

5.4 Comment. Vinogradov in 1990, in an attempt of unification of the Schouten Lie module of multi-vector fields [Schouten 1940, Nijenhuis 1955], with the Frölicher and Nijenhuis Lie-operation, introduced new $\mathbb{R}$-bracket as the sum of double graded commutator of derivations. The value of the Vinogradov binary bracket do not vanish on a ring of the scalars and therefore is not given by the tensor field. Vinogradov proposed the following explicit $\mathbb{R}$-bracket for $A, B \in \operatorname{End}_{\mathcal{F}}(M^\wedge)$

$$2[A \otimes_{\mathbb{R}} B]_V \equiv \{\mathcal{L}_A \otimes_{\mathbb{R}} B\} - (-)^B \{A \otimes_{\mathbb{R}} \mathcal{L}_B\}. \tag{5.20}$$

An evaluation of the Lie-Ślebodziński map gives

$$\mathcal{L}_{[A \otimes_{\mathbb{R}} B]_V} = \{\mathcal{L}_A \otimes_{\mathbb{R}} \mathcal{L}_B\}. \tag{5.21}$$

Contrary to our Definition (5.5) where $[p \otimes_{\mathbb{R}} q] \in M^\wedge \otimes_{\mathcal{F}} M^*$, the Vinogradov bracket do not define a tensor field, $[A \otimes_{\mathbb{R}} B]_V|\mathcal{F} \neq 0$.

5.1 Consequence: modul derivation

The notion of the Leibniz/Loday algebra can be weakened by relaxing the condition of an algebra derivation to a module derivation. De Rham complex $M^\wedge$ with $d \in \operatorname{der}_{\mathbb{R}}(M^\wedge)$ is a DGA. Then an $M^\wedge$-module with a binary operation $[\cdot \otimes_{\mathbb{R}} \cdot]$ is said to be Leibniz/Loday $\mathbb{R}$-algebra if $[\cdot \otimes_{\mathbb{R}} \cdot]$ is $M^\wedge$-module derivation.

5.5 Theorem (e.g. Dubois-Violette and Michor 1994)**.** *Let $p, q \in M^\wedge \otimes_{\mathcal{F}} M^*$ and $\alpha \in M^\wedge$. We abbreviate $\alpha \wedge q$ to αq. The following Leibniz formula for the $M^\wedge$-module graded derivation holds*

$$[p \otimes_{\mathbb{R}} (\alpha q)] = (\mathcal{L}_{ip}\alpha)q - (-)^{p(\alpha+q+1)}\,(d\alpha)(qp) + (-)^{\alpha(p+1)}\,\alpha[p \otimes_{\mathbb{R}} q].$$

The above clue $M^\wedge$-module graded derivation is rather known, however frequently presented without proof. We claim that the proof is a trivial consequence of Definition (5.4)-(5.5). Straightforward calculations using (4.11) proves the above theorem.

Another important easy consequence of Definition (5.4)-(5.5) is the graded Jacobi relation that is an example of the graded Leibniz derivation. With this respect it is instructive to compare with Kanatchikov [1996], where the graded Jacobi relation was derived for 'semi-bracket' $\{i_p \otimes_{\mathbb{R}} \mathcal{L}^r_{iq}\}$, that do not coincide with the Frölicher-Nijenhuis bracket (5.4)-(5.5).

6 Bianchi identity

In this section $p \equiv \tau \otimes_{\mathcal{F}} P \in M \otimes_{\mathcal{F}} M^*$ with $\tau P = 1 \in \mathcal{F}$.

6.1 Zero grade derivation. The composition $i_p = e_\tau \circ i_P \in \operatorname{der}_{\mathcal{F}}(M^\wedge)$ implies $i_P \circ e_\tau | \mathcal{F} = \mathrm{id}_{\mathcal{F}} \cdot \tau P$, and $(i_p)^2 = i_p$,

$$\begin{array}{ccccccccc} M & \xrightarrow{P} & \mathcal{F} & \qquad & \mathcal{F} & \xleftarrow{P} & M & \qquad & M^\wedge & \xrightarrow{i_P} & M^\wedge \\ {\scriptstyle p}\downarrow & & || & & {\scriptstyle \mathrm{id}}\downarrow & & || & & {\scriptstyle p}\downarrow & & || \\ M & \xleftarrow{e_\tau} & \mathcal{F} & & \mathcal{F} & \xrightarrow{e_\tau} & M & & M^\wedge & \xleftarrow{e_\tau} & M^\wedge \end{array} \tag{6.1}$$

However $i_P \circ e_\tau$ does not split on $M^\wedge$, $i_P \circ e_\tau = (\tau P)\,\mathrm{id} - e_\tau \circ i_P \quad \neq \mathrm{id}\,.$

6.2 Angular rotation. Let $p^2 = p \in M^\wedge \otimes_{\mathcal{F}} M^*$, then grade $p = 0$. The angular rotation tensor ω of the $(1,1)$-tensor field p, is defined as follows

$$i_\omega \equiv (\mathrm{id} - i_p) \circ d \circ i_p \quad = -\mathcal{L}_{ip} \circ i_p.$$

We will show that $i_\omega \in \operatorname{der}_{\mathcal{F}}(M^\wedge)$ and therefore ω is a $(2,1)$-tensor field. The name 'angular rotation of idempotent' is motivated in the proof of the next Lemma.

6.3 Theorem (Anholonomy). *Let* $p^2 = p$, $i_p \equiv e_\tau \circ i_P \in \operatorname{der}_{\mathcal{F}}(M^\wedge)$. *Then*

i. $\omega = \frac{1}{2} \cdot [p \otimes_{\mathbb{R}} p]$.
ii. $\omega = (\omega\tau) \otimes_{\mathcal{F}} P = (i_P(\tau \wedge d\tau)) \otimes_{\mathcal{F}} P$.
iii. $i_\omega = \{(d - \tau \wedge \mathcal{L}_{iP}) \otimes_{\mathbb{R}} i_p\}, \quad \{(d - \tau \wedge \mathcal{L}_{iP}) \otimes_{\mathbb{R}} i_P\} = 0$.
iv. $(d - \tau \wedge \mathcal{L}_{iP})^2 = (\omega\tau) \wedge \mathcal{L}_{iP} \simeq$ curvature.

Proof. The proof of (i)-(ii) is straightforward, by direct inspection. The equalities (iii) and (iv) of derivations are a little more involved. The identity (iii), tells that the tensor field ω is 'the spatial divergence' of the connection p, is even more convincing, than adopted Definition 6.2, to interpret ω as the angular rotation tensor field. A two-form $d\tau$ sometimes is called the vortex form of the connection $p \in M \otimes_{\mathcal{F}} M^*$ [Cattaneo]. □

The differential operator, $(e_{\tau/(\tau P)} \circ \mathcal{L}_{iP}) \circ (\mathrm{id} - i_p)$, is invariant with respect to the dilation

$$P \mapsto fP, \quad e_{\tau/(\tau f P)} \circ \mathcal{L}_{fiP} = e_\tau \circ \mathcal{L}_{iP} - e_{df/f} \circ i_p. \tag{6.2}$$

Bianchi identity. Luigi Bianchi introduced his identity in Lezioni di geometria ..., three Volumes published in [1902–1909]. We refer also to [Kolár, Michor and Slovák 1993]. The Bianchi identity for a connection $p \in M \otimes_{\mathcal{F}} M^*$ tells that

$$\tfrac{1}{2}[[p \otimes_{\mathbb{R}} p] \otimes_{\mathbb{R}} p] = [\omega \otimes_{\mathbb{R}} p] = \{\omega \otimes_{\mathbb{R}} (d - \mathcal{L}_{ip})\} = 0. \tag{6.3}$$

7 Frobenius algebra

Let $\mathcal{F}$ denotes an associative and commutative unital ring. Let A be $\mathcal{F}$-module ($\mathcal{F} - \mathcal{F}$-module) and $A^* \equiv \mathrm{mod}_{\mathcal{F}}(A, \mathcal{F})$ be a dual $\mathcal{F}$-module, together with the right and the left evaluations and co-evaluations, also known as the closed/pivotal structures which axioms are given by the Reidemeister zero moves,

$$\begin{array}{ccc} A^* \otimes_{\mathcal{F}} A & \xrightarrow{\text{left evaluation}} & \mathcal{F} \\ A \otimes_{\mathcal{F}} A^* & \xrightarrow{\text{right evaluation}} & \mathcal{F} \\ A^* \otimes_{\mathcal{F}} A & \xleftarrow{\text{left co-evaluation}} & \mathcal{F} \\ A \otimes_{\mathcal{F}} A^* & \xleftarrow{\text{right co-evaluation}} & \mathcal{F} \end{array} \tag{7.1}$$

An $\mathcal{F}$-algebra $m = \curlyvee$ with a Frobenius covector (a co-unit) ε is said to be co-unit-class $\mathcal{F}$-algebra,

$$\begin{aligned} &\curlyvee \in (2 \mapsto 1) \equiv \mathrm{mod}_{\mathcal{F}}(A \otimes_{\mathcal{F}} A, A), \\ &\varepsilon \in (1 \mapsto 0) \equiv \mathrm{mod}_{\mathcal{F}}(A, \mathcal{F}) \equiv A^*. \end{aligned} \tag{7.2}$$

The composition (co-unit $\circ \curlyvee$) is a binary form equivalent to unary left/right $\mathcal{F}$-module map $h^{l/r} \in \mathrm{mod}_{\mathcal{F}}(A, A^*)$,

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\varepsilon \circ m = h^l \circ (\mathrm{ev}^l \otimes \mathrm{id}) = (\mathrm{id} \otimes \mathrm{ev}^r) \circ h^r} & \mathcal{F} \\ A & \xrightarrow{h^l, h^r} & A^* \end{array} \tag{7.3}$$

If a form h^l or/and h^r is non-degenerate, $\ker(h) = 0 \in A$, then $\{m, \varepsilon)\}$ is said to be Frobenius $\mathcal{F}$-algebra [Ferdinand Georg Frobenius (1849-1917), 1903].

An $\mathcal{F}$-co-algebra $\triangle = \curlywedge$ with unit η is said to be unit-class co-algebra,

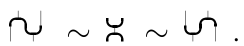
$$\begin{aligned} &\curlywedge \in (1 \mapsto 2) \equiv \mathrm{mod}_{\mathcal{F}}(A, A \otimes_{\mathcal{F}} A), \\ &\eta/1 \in (0 \mapsto 1) \equiv \mathrm{mod}_{\mathcal{F}}(\mathcal{F}, A) \simeq A. \end{aligned} \tag{7.4}$$

The composition $\curlywedge \circ \eta$ is a co-binary form that is equivalent to left/right unary $\mathcal{F}$-module map $f^{l/r} \in \mathrm{mod}_{\mathcal{F}}(A^*, A)$,

$$\begin{array}{ccc} A \otimes_{\mathcal{F}} A & \xleftarrow{\triangle \circ \eta = (f^l \otimes \mathrm{coev}^l) \circ \mathrm{coev}^l = \mathrm{coev}^r \circ (\mathrm{id} \otimes f^r)} & \mathcal{F} \\ A & \xleftarrow{f^l, f^r} & A^* \end{array} \tag{7.5}$$

If this co-binary form $\triangle \circ \eta$ is non-degenerate, $\ker(f^l/f^r) = 0 \in A^*$, then $\{\triangle, (\mathrm{unit} = \eta)\}$ is said to be Frobenius $\mathcal{F}$-co-algebra.

A Frobenius $\mathcal{F}$-algebra is both Frobenius algebra and Frobenius co-algebra subject two Frobenius axioms [Frobenius 1903, Curtis and Reiner 1962, Kauffman 1994, Voronov 1994, Kadison 1999, Caenepeel et al. 2002, Baez 2001],



The Frobenius axioms do not imply uniqueness of $\curlywedge$ for a given model of $\curlyvee$, and vice-versa. The Frobemius axioms can be rephrased as

$$\curlyvee \in \mathrm{bicomod}(||, |), \qquad \curlywedge \in \mathrm{bimod}(|, ||).$$

A Clifford algebra is a particular example of a Frobenius algebra where unary 'handle' $\curlyvee \circ \curlywedge =$  $\in (1 \mapsto 1)$ is diagonal [Oziewicz 2003, Figure 10]. Such Frobenius algebra is also said to be 'canonical'. A Frobenius algebra is antipode-less [Oziewicz 1997, 1998].

7.1 Trace is a co-unit. A trace on $\mathcal{F}$-algebra A is an $\mathcal{F}$-module map, a covector $\mathrm{tr} \in \mathrm{mod}_{\mathcal{F}}(A, F) \equiv A^*$, *i.e.* a co-unit/(Frobenius covector), such that $\forall\, u, v \in A$, $\mathrm{tr}(uv) = \mathrm{tr}(vu)$. An $\mathcal{F}$-algebra A with a trace is said to be trace-class $\mathcal{F}$-algebra.

A unit $\eta \in A \simeq \mathrm{mod}_{\mathcal{F}}(\mathcal{F}, A)$ is said to be co-trace if $\triangle \circ \eta = \triangle^{op} \circ \eta$. An $\mathcal{F}$-co-algebra with co-trace, $\mathrm{cotr} = \mathrm{tr}^*$, is said to be co-trace-class co-algebra,

$$\triangle \circ \mathrm{cotr} = \triangle^{op} \circ \mathrm{cotr}\,. \tag{7.6}$$

The composition $(\mathrm{tr} \circ m)$ is a symmetric binary form, and $(\triangle \circ \mathrm{cotr})$ is a symmetric co-binary form.

The Nijenhuis-Richardson $\mathbb{Z}$-graded $\mathcal{F}$-algebra restricted to zero grade endomorphisms $M \otimes M^*$ is associative and unital trace-clase algebra,

$$M \otimes_{\mathcal{F}} M^* \xrightarrow{\text{trace = counit}} \mathcal{F}, \qquad \mathrm{tr}(pq) = \mathrm{tr}(qp). \tag{7.7}$$

One can extend $\mathcal{F}$-valued trace to $M^\wedge$-valued counit='super-trace' over the Nijenhuis-Richardson nonassociative graded $\mathcal{F}$-algebra

$$M^\wedge \otimes_{\mathcal{F}} M^* \xrightarrow{\text{`trace'}} M^\wedge, \quad \mathrm{tr}(\alpha \otimes_{\mathcal{F}} P) \equiv i_P\alpha \in M^\wedge. \tag{7.8}$$

8 Frobenius subalgebra of Nijenhuis-Richardson algebra

8.1 Definition (Atomic idempotent)**.** An idempotent $p^2 = p \in A$ in an algebra A is said to be an *atom* if $p \wedge (pAp) = 0 \in A^{\wedge 2}$ [Jones, Statistical Mechanics, 1989].

The Nijenhuis-Richardson nonassociative $\mathcal{F}$-algebra possess important associative subalgebra of endomorphisms $\mathrm{End}_{\mathcal{F}} M \equiv \mathrm{mod}_{\mathcal{F}}(M, M)$ (the endomorphism algebra with trivial center is said to be the von Neumann factor). The endomorphism subalgebra is not stable under Frölicher-Nijenhuis Lie differential $\mathbb{R}$-operation, if $p \in \mathrm{End}_{\mathcal{F}} M$ then $[p \otimes_{\mathbb{R}} p] \notin \mathrm{End}_{\mathcal{F}} M$.

We consider unital subalgebra of endomorphism algebra, generated by finite set of *primitive* idempotents (an idempotent $p^2 = p$ is said to be primitive if $p = a + b$ for idempotents a and b with $ab = ba = 0$ imply that $a = 0$ or $b = 0$). It appears that in the generic case such subalgebra 'of idempotents' is Frobenius.

A set $n \in \mathbb{N}$ of primitives idempotents $\{p_1, \ldots, p_n\}$, $\mathrm{tr}(p_i) = 1 \in \mathcal{F}$, and unit u, with a finite trace $\mathrm{tr}\, u = d \in \mathbb{N}$, generate not commutative trace-class Frobenius $\mathcal{F}$-algebra Fr_n (relations are given below) with symmetric form $h \equiv \mathrm{tr} \circ m \in \mathrm{mod}_{\mathcal{F}}((\mathrm{Fr}_n)^{\otimes 2}, \mathcal{F})$. This particular bi-associative and bi-unital/bi-trace Frobenius $\mathcal{F}$-algebra Fr_n is a sub-algebra of Nijenhuis-Richardson algebra, Definition 2.1.

A Frobenius $\mathcal{F}$-algebra Fr_n of atomic/simple idempotents is subject of the following relations,

$$(p_i)^2 = p_i, \quad i = 1, \ldots, n, \tag{8.1}$$

$$\forall\, w \in \mathrm{Fr}_n, \qquad p_i w p_j \ \mathrm{tr}(p_i p_j) = p_i p_j \ \mathrm{tr}(p_i w p_j). \tag{8.2}$$

Every pair of atomic idempotents p and q with $\mathrm{tr}\, p = \mathrm{tr}\, q = 1 \in \mathcal{F}$, satisfy the Galois connection (name introduced by Ore), a property that is also called a generalized inverse

$$pqp = \mathrm{tr}(pq)\, p \quad \text{and} \quad qpq = \mathrm{tr}(pq)\, q. \tag{8.3}$$

This remains the relations of the Jones algebra and of the von Neumann finite dimensional algebra generated by atoms p and q [Jones 1983, §3].

From this it follows that a length of every word in Frobenius $\mathcal{F}$-algebra Fr_n must be ≤ 2, and the $\mathcal{F}$-dimensions are

$$\dim_{\mathcal{F}}(\mathrm{Fr}_n) = 1 + n^2 \quad = 1, 2, 5, 10, 17, 26, \ldots$$

8.2 Theorem (Laplace expansion)**.** *The Frobenius covector is given by a trace tr* $\in (\mathrm{Fr})^*$*. The following Laplace expansion holds, also called 'weak coalgebra' condition. In the Sweedler notation for three words* $a, b, c \in \mathrm{Fr}_n$),

$$\mathrm{tr}(abc) = \Sigma\ \mathrm{tr}(a_1 c)\, \mathrm{tr}(a_2 b). \tag{8.4}$$

In particular for $a = b = c = u \equiv \eta$,

$$\mathbb{N} \ni d \equiv \mathrm{tr} \circ \mathrm{cotr} \equiv \mathrm{tr}(u) = \Sigma\ \mathrm{tr}(u_1)\, \mathrm{tr}(u_2), \tag{8.5}$$

$$\curlywedge u \neq u \otimes u \xmapsto{\curlyvee} u \xmapsto{\mathrm{tr}} d. \tag{8.6}$$

8.3 Theorem (Frobenius coalgebra)**.** *Let* $\{e_i \in \mathrm{Fr}_n\}$ *be a basis diagonalizing* $h = \mathrm{tr} \circ \curlyvee$*, i.e.* $h(e_i \otimes e_j) \equiv \mathrm{tr}(e_i e_j) = h_i \delta_{ij}$*. Then*

$$\curlywedge\ e_i = \mathrm{tr}(e_i e_k e_l) \frac{e_l}{h_l} \otimes \frac{e_k}{h_k}.$$

The Frobenius algebra of atomic idempotents is antipode-less.

9 Frobenius algebra of two idempotents

The bilinear form on 2-dimensional $\mathcal{F}$-algebra $\mathrm{Fr}_1 = \mathrm{span}_{\mathcal{F}}\{u,p\}$ for $1<d$ is positive definite $(++)$. To see this, let choose the volume form as $z_1 \equiv u\wedge p \in (\mathrm{Fr}_1)^{\wedge 2}$. Then $\det_z h \equiv (h^\wedge z)z = d-1$. The form $h \equiv \mathrm{tr}\circ\curlyvee$ in the basis $\{u,p\}$ and in the basis $\{u-p,p\}$ (after Gram-Schmidt orthogonalization) possess the following basis-dependent matrix presentations

$$h\begin{pmatrix}u\\p\end{pmatrix}=\begin{pmatrix}d&1\\1&1\end{pmatrix}\begin{pmatrix}u^*\\p^*\end{pmatrix},\quad h\begin{pmatrix}u-p\\p\end{pmatrix}=\begin{pmatrix}d-1&0\\0&1\end{pmatrix}\begin{pmatrix}u^*\\u^*+p^*\end{pmatrix}. \tag{9.1}$$

A coalgebra Fr_1 is group-like (no no-zero primitives). The co-unital comultiplication is not unital

$$\curlywedge(u-p)=\frac{(u-p)\otimes(u-p)}{\mathrm{tr}(u-p)},\quad \curlywedge p = p\otimes p. \tag{9.2}$$

The Lagrange/Sylvester theorem and Gram-Schmidt orthogonalization allows to calculate the signature for Frobenius $\mathcal{F}$-algebra Fr_n for any $n\in\mathbb{N}$. Here we wish to report signature for five-dimensional Frobenius $\mathcal{F}$-algebra $\mathrm{Fr}_2 = \mathrm{gen}\{p,q\}$ generated by two atomic idempotents.

9.1 Theorem (Signature). *Let $t\equiv \mathrm{tr}(pq)\neq\{-1,0,+1\}$. The signature of the bilinear form $h\simeq \mathrm{tr}\circ m : (\mathrm{Fr}_2)^{\otimes 2}\longrightarrow\mathcal{F}$ for five-dimensional $\mathcal{F}$-algebra* Fr_2, $\dim_{\mathcal{F}}(\mathrm{Fr}_2)=5$, *depends on $d\equiv\mathrm{tr}(u)\in\mathbb{N}$ only.*

$$\text{Signature } \textit{of } h = \begin{cases} -++++ & \textit{if}\quad d>2,\\ -+++0 & \textit{if}\quad d=2,\\ -+++- & \textit{if}\quad d<2.\end{cases}$$

Proof. Let p and $q\in\mathrm{Fr}_2$ be generating atomic idempotents. The center $Z\,\mathrm{Fr}_2$ of Frobenius $\mathcal{F}$-algebra Fr_2 is two-dimensional,

$$\dim_{\mathcal{F}}(Z\,\mathrm{Fr}_2)=2,\qquad u,(p-q)^2\in Z\,\mathrm{Fr}_2, \tag{9.3}$$
$$(pq+qp)^2=t(p+q)^2,\quad (p-q)^4=-(t-1)(p-q)^2. \tag{9.4}$$

Let a volume form for a $\mathcal{F}$-module Fr_2 be $z_2\equiv u\wedge p\wedge q\wedge pq\wedge qp\in(\mathrm{Fr}_2)^{\wedge 5}$. Then $\det_z(\mathrm{tr}\circ m)=-(d-2)(t-1)^4t^2$. In the basis $\{u,p,q,pq,qp\}$ the bilinear form $h \equiv \mathrm{tr} \circ m$ has the following basis-dependent-matrix

$$h\begin{pmatrix} u \\ p \\ q \\ pq \\ qp \end{pmatrix} = \begin{pmatrix} d & 1 & 1 & t & t \\ 1 & 1 & t & t & t \\ 1 & t & 1 & t & t \\ t & t & t & t^2 & t \\ t & t & t & t & t^2 \end{pmatrix} \begin{pmatrix} u^* \\ p^* \\ q^* \\ (pq)^* \\ (qp)^* \end{pmatrix} \tag{9.5}$$

For $t \neq \{-1, 0, +1\}$, the particular basis of Fr_2 diagonalizing the form $h = \mathrm{tr} \circ \mathrm{cotr}$ is

$$u + \frac{(p-q)^2}{t-1}, \quad qp, \quad p + tq - (pq + qp), \quad q - \frac{pq + qp}{t+1}, \quad pq - \frac{qp}{t}. \tag{9.6}$$

In this basis the matrix of the scalar product h is diagonal,

$$h \simeq \mathrm{diag}\left(d - 2, t^2, (t-1)^2, -\frac{t-1}{t+1}, t^2 - 1\right). \quad \square \tag{9.7}$$

10 Conclusion

The Frölicher and Nijenhuis Lie $\mathbb{R}$-algebra structure on universal Graßmann-module of differential multi-forms found increasing number of important applications/interpretations both in pure algebra and in differential geometry of Ehresmann connections [Kocik 1997, Wagemann 1998], as well as in many branches of mathematical physics, in the special and in the general theory of relativity [Minguzzi 2003], in Maxwell's theory of electromagnetic field [Fecko 1997, Kocik 1997, Cruz and Oziewicz 2003], in Hamilton-Jacobi theory in classical mechanics [Gruhn and Oziewicz 1983], in symplectic geometry of the Lagrangian and Hamiltonian mechanics [Chavchanidze 2003], etc.

From the point of view of these numerous fundamental applications there is a need for the algorithmic computational programming methods to deals with many structural aspects of this non trivial Lie $\mathbb{R}$-algebra. The present paper was motivated by this need of explicit/algorithmic easy to handle definition of the Frölicher and Nijenhuis Lie operation. We are proposing here such definition of the Frölicher and Nijenhuis Lie operation (5.4)-(5.5). This definition has a clear advantage that can be implemented for computational symbolic program in computer algebra.

Many identities that hold in Frölicher and Nijenhuis Lie Graßmann-module follows much easily from proposed definition.

It is important that the Definition (5.4)-(5.5) of Lie $\mathbb{R}$-algebra needs nonassociative Frölicher-Richardson $\mathcal{F}$-operation on universal Graßmann-module. The Frölicher-Richardson nonassociative $\mathcal{F}$-algebra deserve future studies in many respects. The Frölicher-Richardson algebra include associative endomorphism subalgebra. Of special interests, from fundamental physical theories, quantum mechanics and relativity theory, are endomorphism subalgebras generated by atomic idempotents. Such generic subalgebras are Frobenius algebras, they possess non-degenerate scalar product that gives antipode-less algebra structure. In the last Sections the Frobenius algebra is illustrated on example of the five-dimensional algebra generated by two atomic idempotents. We believe that the correct environment for these particular Frobenius associative algebras must be nonassociative Frölicher-Richardson algebra, because the Frölicher and Nijenhuis differential Lie operation do not preserve associative endomorphism algebra. If $p \in M \otimes_{\mathcal{F}} M^*$ is an endomorphism, then the Frölicher and Nijenhuis differential Lie operation (5.4)-(5.5) gives $[p \otimes_{\mathbb{R}} p] \notin M \otimes_{\mathcal{F}} M^*$, but $[p \otimes_{\mathbb{R}} p]$ is inside the Frölicher-Richardson algebra. We conjecture that the Frobenius associative algebra could be related/identified with the kinematics and the Frölicher-Richardson not associative algebra with dynamics,

Kinematics	Dynamics
Special relativity	General relativity Gravity
Frobenius algebra	Frölicher-Richardson algebra

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